

# A Generalization of the maximal-spacings in several dimensions and a convexity test.

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## Abstract

The notion of maximal-spacing in several dimensions was introduced and studied by Deheuvels (1983) for data uniformly distributed on the unit cube. Later on, Janson (1987) extended the results to data uniformly distributed on any bounded set, and obtained a very fine result, namely, he derived the asymptotic distribution of different maximal-spacings notions. These results have been very useful in many statistical applications.

We extend Janson's results to the case where the data are generated from a Hölder continuous density that is bounded from below and whose support is bounded. As an application, we develop a convexity test for the support of a distribution.

*Key words:* maximal spacing, convexity test, non-parametric density estimation.

## 1 Introduction

The notion of spacings, which for one dimensional data are just the differences between two consecutive order statistics, have been extensively studied in the one dimensional setting; see, e.g., the review papers [18, 19]. Many important applications to testing and estimation problems have been derived from the study of the asymptotic behaviour of the spacings. Applications to testing problems date back to [20], who address the asymptotic theory of a class of tests for Increasing Failure Rate. For estimation problems, [21] propose the maximum spacing estimation method to estimate the parameters of a univariate statistical model.

In the multidimensional case, several different notions of maximal-spacing have been proposed. Most of them are based on the nearest-neighbors balls (see for instance [14] or [2]) or on the Voronoi tessellation [16], (a comparison can be found in [22]), but they do not capture the key idea of 'largest set missing the observations'. In contrast, this is the case with the different and global notion proposed in [7], and generalized in [12].

In [7], the notion of maximal-spacing is defined and studied for iid data uniformly distributed in  $[0, 1]^d$  as the maximal length  $a$  of a cube  $C = \prod [x_i, x_i + a]$ , included in  $[0, 1]^d$  that does not contain any of the observations. This notion has been extended in [12], in which the uniformity assumption remains but the support of the distribution is no longer assumed to be  $[0, 1]^d$  but may be any compactum  $S$ . Moreover,  $C$  is allowed to be any compact and convex set. Finally, while in [7] only bounds are given, in [12] the asymptotic distribution for the maximal spacing is provided.

The notion of maximal multivariate spacing, and in particular Janson's result, has been used to solve different statistical problems. In set estimation (see, for instance, [3]

and [4] ), it is used to prove the optimality of the rates of convergence.

The aim of this paper is to extend Janson's result to Hölder continuous densities, and develop, using that extension, a test to decide whether the support is convex or not. It is organized as follows: Section 2 is devoted to the extension of Janson's results. A new definition (which includes Janson's as a particular case) is given and the associated theoretical results are presented. The proofs of these results are given in Appendix A. Section 3 is dedicated to the problem of testing the convexity of the support. The corresponding proofs are given in Appendix B. Just to mention some application of this test, let us recall that when dealing with support estimation, if the support is known to be convex, the convex hull of the observations provides a consistent and well studied (see for instance [23],[24] and [27]) estimator of  $S$  which does not require any smoothing parameter. Also, a convexity test can be used to, a posteriori, select a tuning parameter. In [6] a test for convexity is also proposed, and applied to choose the parameter of the ISOMAP (see [26]) method for dimensionality reduction. The convexity test based on Janson's extension allows us to provide an estimation of the  $p$ -value, whereas, in [6], it has to be estimated via the Monte Carlo method.

## 2 Main definitions and results

We start by fixing some notation that will be used throughout the paper.

Given a set  $S \subset \mathbb{R}^d$ , we denote by  $\partial S$ ,  $\hat{S}$ ,  $\bar{S}$ ,  $\text{diam}(S)$ ,  $|S|$ , and  $\mathcal{H}(S)$ , the boundary, interior, closure, diameter, Lebesgue measure, and convex hull of  $S$ , respectively. We denote by  $\|\cdot\|$  and  $\langle \cdot, \cdot \rangle$  the euclidean norm and the inner product respectively. We write  $N(S, \varepsilon)$  for the inner covering number of  $S$  (i.e.: the minimum number of balls of radius  $\varepsilon$  and centred in  $S$  required to cover  $S$ ). Recall that if  $S$  is compact, there exists  $C_S$  such that  $N(S, \varepsilon) \leq C_S \varepsilon^{-d}$ . We denote by  $\mathcal{B}(x, \varepsilon)$  the closed ball in  $\mathbb{R}^d$ , of radius  $\varepsilon$ , centered at  $x$ . We set  $\omega_d = |\mathcal{B}(0, 1)|$ . Given  $\lambda \in \mathbb{R}$ ,  $A, C \subset \mathbb{R}^d$ , we set  $\lambda A = \{\lambda a : a \in A\}$ ,  $A \oplus C = \{a + c : a \in A, c \in C\}$ , and  $A \ominus C = \{x : \{x\} \oplus C \subset A\}$ . For the sake of simplicity, we use the notation  $x + C$ , instead of  $\{x\} \oplus C$ . If  $\lambda \geq 0$ , we set  $A^\lambda = A \oplus \lambda \mathcal{B}(0, 1)$  and  $A^{-\lambda} = A \ominus \lambda \mathcal{B}(0, 1)$ . Given  $A, C \subset \mathbb{R}^d$  two non-empty compact sets, the Hausdorff (or Pompeiu–Hausdorff) distance between them is given by

$$d_H(A, C) = \max \left\{ \max_{a \in A} d(a, C), \max_{c \in C} d(c, A) \right\},$$

where  $d(a, C) = \inf\{\|a - c\| : c \in C\}$ .

Let  $A \subset \mathbb{R}^d$  be a compact convex set with  $|A| = 1$ , let  $v$  be a vector of  $\mathbb{R}^d$ , and let  $\alpha_A(v)$  be the constant defined in equation (2.4) of [12]:

$$\alpha_A(v) = \frac{1}{d!} \int \dots \int |\text{Det}(n(y_i))_{i=1}^d| d\omega(y_1) \dots d\omega(y_d), \quad (1)$$

where  $\omega$  denotes the  $d-1$  dimensional Hausdorff measure, and for  $y \in \partial A$ ,  $n(y)$  denotes the exterior unit normal vector to  $A$  at  $y$ . The integral in (1) is over all  $y_1, \dots, y_d \in \partial A$

such that  $v$  is a linear combination of  $n(y_1), \dots, n(y_d)$  with positive coefficients, and  $\text{Det}(n(y_i))_{i=1}^d$  is the determinant of the vectors  $n(y_i)$  in an orthonormal basis. Corollary 7.4 in [13] proves that  $\alpha_A(v)$  is almost everywhere independent of  $v$ , so that there can be defined an  $\alpha_A$  such that  $\alpha_A(v) = \alpha_A$  almost everywhere.

If  $A$  is the unit cube, then  $\alpha_A = 1$ ; while if  $\mathcal{B}$  is the unit ball, then

$$\alpha_{\mathcal{B}} = \frac{1}{d!} \left( \frac{\sqrt{\pi} \Gamma(\frac{d}{2} + 1)}{\Gamma(\frac{d+1}{2})} \right)^{d-1}.$$

Lastly, let  $U$  be a random variable such that  $\mathbb{P}(U \leq t) = \exp(-\exp(-t))$ .

## 2.1 Janson's result and its extension

Let  $S \subset \mathbb{R}^d$  be a bounded set with  $|S| = 1$  and  $|\partial S| = 0$ . Let  $\aleph_n = \{X_1, \dots, X_n\}$  be iid random vectors uniformly distributed on  $S$ , and  $A$  a bounded convex set. In [12], the maximal-spacing is defined as

$$\Delta^*(\aleph_n) = \sup \left\{ r : \exists x \text{ such that } x + rA \subset S \setminus \aleph_n \right\}.$$

To generalize the results of [12] to the non-uniform case, we need to extend the definition of maximal-spacing. When the sample is drawn according to a probability measure  $\mathbb{P}_X$ , we consider the probability measure of the largest set  $\lambda A$  missing  $\aleph_n$ . If  $|A| = 1$ ,  $\Delta^*(\aleph_n)^d$  is the Lebesgue measure of the largest set  $x + rA \subset S \setminus \aleph_n$ . When the sample is drawn from a non-uniform probability measure, is natural to use the same definition, replacing the Lebesgue measure by the true underlying distribution  $\mathbb{P}_X$ . If  $\mathbb{P}_X$  has continuous density  $f$ , then  $\mathbb{P}_X(x + rA) \sim f(x)r^d$  for sufficiently small  $r$ , so one can define the maximal-spacing as the largest  $r$  such that there exists  $x$  with  $x + \frac{r}{f(x)^{1/d}}A \subset S \setminus \aleph_n$ .

**Definition 1.** Let  $\aleph_n = \{X_1, \dots, X_n\}$  be an iid random sample of points in  $\mathbb{R}^d$ , drawn according to a density  $f$  with bounded support  $S$ . Let  $A \subset \mathbb{R}^d$  be a convex and compact set such that  $|A| = 1$  and its barycentre is the origin of  $\mathbb{R}^d$ . We define

$$\Delta(\aleph_n) = \sup \left\{ r : \exists x \text{ such that } x + \frac{r}{f(x)^{1/d}}A \subset S \setminus \aleph_n \right\},$$

$$V(\aleph_n) = \Delta^d(\aleph_n),$$

and

$$U(\aleph_n) = n\Delta^d(\aleph_n) - \log(n) - (d-1)\log(\log(n)) - \log(\alpha_A).$$

The following result can be found in [12].

**Theorem 1.** Let  $S \subset \mathbb{R}^d$  be a bounded set such that  $|S| = 1$  and  $|\partial S| = 0$ . Let  $\aleph_n = \{X_1, \dots, X_n\}$  be iid random vectors uniformly distributed on  $S$ . Then,

i)

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \text{when } n \rightarrow \infty,$$

ii)

$$\liminf_{n \rightarrow +\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} = d - 1 \text{ a.s.},$$

iii)

$$\limsup_{n \rightarrow +\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} = d + 1 \text{ a.s.}$$

A rescaling extends these results to the case where  $|S| \neq 1$ .

**Corollary 1.** *Let  $S \subset \mathbb{R}^d$  be a bounded set such that  $|\partial S| = 0$  and  $|S| > 0$ . Let  $\aleph_n = \{X_1, \dots, X_n\}$  be iid random vectors uniformly distributed on  $S$ . Then,*

i)

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \text{when } n \rightarrow \infty,$$

ii)

$$\liminf_{n \rightarrow +\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} = d - 1 \text{ a.s.},$$

iii)

$$\limsup_{n \rightarrow +\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} = d + 1 \text{ a.s.}$$

Janson's result does not require any condition on the shape of the support  $S$ , while in our extension it will be required that the inside covering number of  $\partial S$  is such that there exists  $C_{\partial S} > 0$  and  $\kappa < d$  satisfying  $N(\partial S, \epsilon) \leq C_{\partial S} \epsilon^{-\kappa}$ . Note that this is a very mild hypothesis: if  $\partial S$  is smooth enough (for instance, a  $\mathcal{C}^1$   $(d-1)$ -dimensional manifold), it is fulfilled for  $\kappa = d-1$ . More generally, it also holds for any set  $S$  with finite Minkowski content of the boundary, for  $\kappa = d-1$  (see, for instance [15]). With respect to the distribution of the sample, we require that the density is Hölder continuous on  $S$  (i.e. there exists  $K_f$  and  $\beta \in (0, 1]$  such that for all  $x, y \in S$ ,  $|f(x) - f(y)| \leq K_f \|x - y\|^\beta$ ) and bounded from below on its support, by a positive constant  $f_0$ .

This is stated in our main theorem, given below.

**Theorem 2.** *Let  $\aleph_n = \{X_1, \dots, X_n\}$  be iid random vectors distributed according to a distribution  $\mathbb{P}_X$  whose density  $f$  with respect to Lebesgue measure is Hölder continuous and bounded from below on its support  $S$ . Let us assume that  $S$  is compact, and there exists  $\kappa < d$  and  $C_{\partial S} > 0$  such that  $N(\partial S, \epsilon) \leq C_{\partial S} \epsilon^{-\kappa}$ . Then, we have that*

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \text{when } n \rightarrow \infty, \tag{2}$$

$$\liminf_{n \rightarrow +\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} \geq d - 1 \text{ a.s.}, \tag{3}$$

$$\limsup_{n \rightarrow +\infty} \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} \leq d + 1 \text{ a.s.} \quad (4)$$

The proof is given in Appendix A.

### 3 A new test for convexity

#### 3.1 The semi-parametric case

In this section we propose, using the concept of maximal-spacing defined in Section 2, a consistent hypothesis test based on an iid sample  $\{X_1, \dots, X_n\}$  uniformly distributed on a compact set  $S$ , to decide whether  $S$  is convex or not.

The main idea is the following: if  $S$  is convex and the sample is uniformly distributed on  $S$ , then  $\mathcal{H}(\aleph_n)$  is a good approximation to  $S$  and  $|\mathcal{H}(\aleph_n)|^{-1} \mathbb{I}_{\mathcal{H}(\aleph_n)}$  is a good approximation of the uniform law. As a result,

$$\tilde{\Delta}(\aleph_n) = \sup \left\{ r : \exists x \text{ such that } x + r|\mathcal{H}(\aleph_n)|^{1/d} \mathcal{B}(0, 1) \subset \mathcal{H}(\aleph_n) \setminus \aleph_n \right\}$$

is a plug-in estimator of the maximal spacing and should converge to 0. On the other hand, if  $S$  is not convex,  $\tilde{\Delta}(\aleph_n)$  is expected to converge to a positive constant (that depends on the shape of  $S$ ). In order to unify notation, let us first define the maximal inner radius.

**Definition 2.** Let  $S \subset \mathbb{R}^d$  be a bounded set satisfying  $\mathring{S} \neq \emptyset$ . We define the maximal inner radius of  $S$  as

$$\mathcal{R}(S) = \sup \left\{ r : \exists x \in S \text{ such that } \mathcal{B}(x, r) \subset S \right\}.$$

**Remark 1.** We have  $\Delta(\aleph_n) = \mathcal{R}(S \setminus \aleph_n) \omega_d^{1/d} |S|^{1/d}$  and  $\tilde{\Delta}(\aleph_n) = \mathcal{R}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \omega_d^{1/d} |\mathcal{H}(\aleph_n)|^{1/d}$ .

When testing the convexity of the support using a test statistic based on  $\tilde{\Delta}(\aleph_n)$ , we only obtain, in general, an upper asymptotic bound on the test level. However, if the boundary of the support is smooth enough, we have a converging estimation of the level. The regularity condition is the following.

**Condition (P):** For all  $x \in \partial S$  there exists a unique vector  $\xi = \xi(x)$  with  $\|\xi\| = 1$ , such that  $\langle y, \xi \rangle \leq \langle x, \xi \rangle$  for all  $y \in S$ , and

$$\|\xi(x) - \xi(y)\| \leq l \|x - y\| \quad \forall x, y \in \partial S,$$

where  $l$  is a constant. We will denote by  $\mathcal{C}_P$  the class of convex subsets that satisfy condition (P).

The convexity test and its asymptotic behaviour is given in the following theorem.

**Theorem 3.** *Let  $S \subset \mathbb{R}^d$  be compact with non-empty interior. Let  $\aleph_n = \{X_1, \dots, X_n\}$  be a set of iid random vectors uniformly distributed on  $S$ . For the following decision problem,*

$$\begin{cases} H_0 : & \text{the set } S \text{ is convex} \\ H_1 : & \text{the set } S \text{ is not convex,} \end{cases} \quad (5)$$

*the test based on the statistic  $\tilde{V}_n = |\mathcal{H}(\aleph_n)|\omega_d \mathcal{R}(\mathcal{H}(\aleph_n) \setminus \aleph_n)^d$  with critical region given by*

$$RC = \{\tilde{V}_n > c_{n,\gamma}\},$$

*where*

$$c_{n,\gamma} = \frac{1}{n} \left( -\log(-\log(1-\gamma)) + \log(n) + (d-1)\log(\log(n)) + \log(\alpha_{\mathcal{B}}) \right),$$

*and  $\alpha_{\mathcal{B}}$  is the constant defined in (1), is asymptotically of level less than or equal to  $\gamma$ . Moreover, if  $S \in \mathcal{C}_P$ , the asymptotic level is  $\gamma$ . If  $S$  is not convex, the test has power one for all sufficiently large  $n$ .*

The proof of Theorem 3 is given in Appendix B.

### 3.2 The non-parametric case

We now assume that we have a sample  $\aleph_n = \{X_1, \dots, X_n\}$  of iid random vectors in  $\mathbb{R}^d$  drawn according to an unknown density  $f$ . As in the semi-parametric case, the idea is to estimate the maximal-spacing and use this estimation as a test statistic. As before,  $\mathcal{H}(\aleph_n)$  is proposed as an estimator of  $S$ . To ensure that the test proposed in Theorem 3 allows determining whether the support is convex or not, the density estimator should have a non-conventional behaviour: it is expected to converge toward the unknown density when the support is convex, but not when the support is not convex. That is why we propose the following density estimator.

**Definition 3.** *Let  $\text{Vor}(X_i)$  be the Voronoi cell of the point  $X_i$  (i.e.  $\text{Vor}(X_i) = \{x : \|x - X_i\| = \min_{y \in \aleph_n} \|x - y\|\}$ ). If  $K$  is a kernel function (i.e.  $K \geq 0$ ,  $\int K = 1$  and  $\int uK(u)du = 0$ ) and  $f_n(x) = \frac{1}{nh_n^d} \sum K((x - X_i)/h_n)$  denotes the usual kernel density estimator, we define*

$$\hat{f}_n(x) = \max_{i: x \in \text{Vor}(X_i)} f_n(X_i) \mathbb{I}_{x \in \mathcal{H}(\aleph_n)}. \quad (6)$$

We propose to test the convexity using the following plug-in estimator of  $\Delta(\aleph_n)$ :

$$\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) = \sup \left\{ r : \exists x \text{ such that } x + \frac{r}{\hat{f}_n(x)^{1/d}} A \subset \mathcal{H}(\aleph_n) \setminus \aleph_n \right\},$$

with  $A = \omega_d^{-1/d} \mathcal{B}(0, 1)$ , and reject  $H_0$  (the support is convex) if  $\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n)$  is sufficiently large.

The proof of Theorem 4 makes use of Theorem 2.3 in [11]. In order to apply that result, we will introduce some technical hypotheses on the kernel function.

**Definition 4.** Let  $\mathcal{K}$  be the set of kernel functions  $K(u) = \phi(p(u))$ , where  $p$  is a polynomial and  $\phi$  a bounded real function of bounded variation, such that  $c_K = \int \|u\| K(u) du < \infty$ ,  $K \geq 0$  and there exists  $r_K$  and  $c'_K > 0$  such that  $K(x) \geq c'_K$  for all  $x \in \mathcal{B}(0, r_K)$ .

Note that, for example, the Gaussian and the uniform kernel are in  $\mathcal{K}$ .

**Definition 5.** A set  $S$  is standard if there exist positive numbers  $r_0, c_S$  such that  $|\mathcal{B}(x, r) \cap S| \geq c_S \omega_d r^d$  for all  $r \leq r_0$ . We write  $\mathcal{C}$  for the class of compact convex sets with non-empty interior, and  $\mathcal{A}$  for the class of all compact standard sets.

Finally it is also necessary to impose some conditions on the density.

**Condition (B):** A density  $f$  with support  $S$  fulfils condition B if its restriction to  $S$  is Lipschitz continuous (i.e. there exists  $k_f$  such that  $\forall x, y \in S, |f(x) - f(y)| \leq k_f \|x - y\|$ ) and there exists  $f_0 > 0$  such that  $f(x) \geq f_0$  for all  $x \in S$ . We denote  $f_1 = \max_{x \in S} f(x)$ .

**Remark 2.** The condition  $f(x) \geq f_0 > 0$  for all  $x \in S$  is a necessary condition to test convexity, as indeed is mentioned in [6]: ‘...an assumption like the density being bounded away from zero on its support is necessary for consistent decision rules.’

**Theorem 4.** Let  $K \in \mathcal{K}$ , and let  $\hat{f}_n$  be as defined in (3). Assume that  $h_n = \mathcal{O}(n^{-\beta})$  for some  $0 < \beta < 1/d$ . Assume also that the unknown density fulfils condition B. For the following decision problem,

$$\begin{cases} H_0 : & S \in \mathcal{C} \\ H_1 : & S \notin \mathcal{C}, \end{cases} \quad (7)$$

a) the test based on the statistic  $\hat{V}_n = \hat{\delta}(\mathcal{H}(\mathfrak{N}_n) \setminus \mathfrak{N}_n)^d$  with critical region  $RC = \{\hat{V}_n \geq c_{n,\gamma}\}$ , where

$$c_{n,\gamma} = \frac{1}{n} \left( -\log(-\log(1 - \gamma)) + \log(n) + (d - 1) \log(\log(n)) + \log(\alpha_B) \right),$$

has an asymptotic level less than  $\gamma$ .

b) Moreover, if  $S \in \mathcal{A}$  is not convex, the power is 1 for sufficiently large  $n$ .

**Remark 3.** Notice that the ‘optimal’ kernel sequence size,  $h_n = h_0 n^{1/(d+4)}$ , satisfies the hypothesis of our theorem, so that any bandwidth selection method should be suitable for testing for convexity.

However, in the semi-parametric case it is possible to derive the asymptotic behaviour for the level under regularity conditions on the support. In this more general setup, we will not have a convergent level estimation but only a bound for the level (the price to pay for estimating the density). The proof of Theorem 4 is given in Appendix B.

### 3.3 Simulations

We have performed two simulation studies to assess the behaviour of our test in the scenarios described in Sections 3.1 and 3.2. For the first study, the data were drawn uniformly from sets  $S \subset \mathbb{R}^2$ , and we will perform the test defined in Section 3.1 to obtain estimations of the power and the level. In the second study, the non-parametric case, the data can be not uniformly drawn, and we estimate the density using the estimator given by (6). In this case, we consider the same sets and density as in [6].

#### 3.3.1 Semi-Parametric case

The data were generated uniformly from the sets  $S_\varphi = [0, 1]^2 \setminus T_\varphi$ , where  $T_\varphi$  is the isosceles triangle with height 1/2 (see Figure 1) whose angle at the vertex (1/2, 1/2) is equal to  $\varphi$ . If we have a random sample from  $S_\varphi$ , it is clear that as  $\varphi$  increases, it should be easier to detect the non-convexity of the set. The results of the simulations are summarized in Table 1.

$\varphi = \pi/4$		$\varphi = \pi/6$		$\varphi = \pi/8$	
n	$\hat{\beta}$	n	$\hat{\beta}$	n	$\hat{\beta}$
100	.4	200	.565	300	.543
130	.636	250	.787	350	.679
160	.835	300	.926	400	.846
200	.946	400	.996	500	.976
300	.997	500	1	600	.997

Table 1: Power estimated over 1000 replications, for different values of  $\varphi$ , when the sample is uniformly distributed on  $[0, 1]^2 \setminus T_\varphi$ , where  $T_\varphi$  is an isosceles triangle, (see Figure 1).

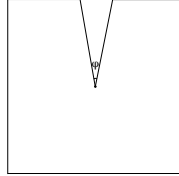


Figure 1:  $[0, 1]^2 \setminus T_\varphi$  where  $T_\varphi$  is an isosceles triangle with height 1/2.

#### 3.3.2 Non-parametric case

We performed a simulation study for the same sets used in [6]. Consider the curves  $\gamma_{R,\theta} = R(\cos(\theta), \sin(\theta))$  with  $\theta \in [\frac{3\pi(R-1)}{2R}, \frac{3}{2}\pi]$  and the reflections of those curves along the  $y$  axis (which will be denoted by  $\zeta_{R,\theta}$ ). We consider  $\Gamma_R = T_{(0,R)}(\gamma_{R,\theta}) \cup T_{(0,-R)}(\zeta_{R,\theta})$  with  $\theta \in [\frac{3\pi(R-1)}{2R}, \frac{3}{2}\pi]$ , where  $T_v$  is the translation along the vector  $v$ . It is easy to



see that the length of every  $\Gamma_R$  is  $\frac{3}{2}\pi$ . We will consider, for different values of  $R$ , the  $S$ -shaped sets (see the first row in Figure 2)

$$S_R = T_{(0,R)} \left( \bigcup_{R-0.6 \leq r \leq R+0.6} \gamma_{r,\theta} \right) \cup T_{(0,-R)} \left( \bigcup_{R-0.6 \leq r \leq R+0.6} \zeta_{r,\theta} \right).$$

Observe that when  $R$  approaches infinity, the sets  $S$  converge to a rectangle (which corresponds to the convex case). We have generated the data according to two different densities. The first one is the same as that considered in [6]: that is, along the orthogonal direction of  $\Gamma_R$ , we choose a random variable with normal density (with zero mean and standard deviation  $\sigma = 0.15$ ) truncated to 0.6 (the truncation is performed to ensure that we obtain a point in the set  $S_R$ ). In the second case, we consider a random variable along the orthogonal direction of  $\Gamma_R$  but uniformly distributed on  $[-0.6, 0.6]$ . In Tables 2 and 3, we have summarized the results of the simulations, for different sample sizes (we performed the test  $B = 100$  times). The results are quite encouraging and slightly better than those obtained in [6] since the non-convexity is better detected (see Fig. 7 in [6] for comparison) with no need for the decision rule to be calibrated.

R	N=100		N=250		N=500		N=1000	
	np	unif	np	unif	np	unif	np	unif
1	.13	.44	.55	.99	1	1	1	1
1.5	.98	1	1	1	1	1	1	1
3	.38	.24	1	1	1	1	1	1
6	.08	.09	.41	.66	1	1	1	1
12	.01	.05	.02	.08	.39	.68	.98	1
24	0	.07	.01	.05	0	.09	.07	.48
$\infty$	0	.04	0	.09	0	.04	.01	.05

Table 2: Power estimated over  $B$  replications, for different values of  $R$ , when the sample is uniformly distributed along the orthogonal direction of  $\Gamma_R$ .

R	N=100		N=250		N=500		N=1000	
	np	unif	np	unif	np	unif	np	unif
1	1	1	1	1	1	1	1	1
1.5	1	1	1	1	1	1	1	1
3	1	.99	1	1	1	1	1	1
6	.67	.41	.99	1	1	1	1	1
12	.25	.19	.62	.98	.85	1	.94	1
24	.1	.30	.30	.92	.38	1	.48	1
$\infty$	0	.33	.04	.92	.06	1	.04	1

Table 3: Power estimated over  $B$  replications, for different values of  $R$ , when the sample is drawn according to a truncated normal distribution along the orthogonal direction of  $\Gamma_R$ .

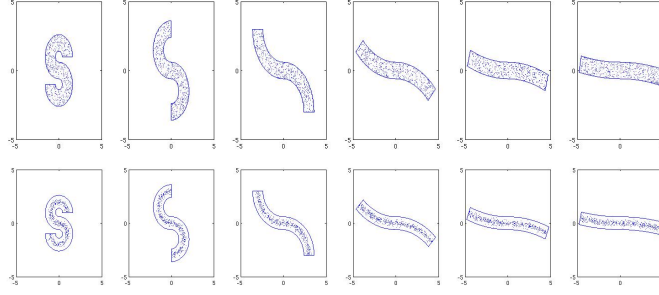


Figure 2:  $S_R$  for different values of  $R$  together with the sample drawn with a uniform radial noise (top) and with a truncated Gaussian noise (bottom)

## 4 Appendix A

In Appendix A we proof the main result on the generalization of the maximal-spacing, given in Theorem 2. First we settle some preliminary lemmas, then we prove a weaker version of Theorem 2, for the case of piecewise constant densities on disjoint sets. We continue by considering piecewise constant densities, and finally we prove the result for Hölder continuous densities.

### 4.1 Preliminary Lemmas

As we mentioned before, the proof of Corollary 1 follows from a simple rescaling in the lemmas stated in [12], used to prove Theorem 1. In particular the following rescaled lemma will be used in this section.

**Lemma 1.** *Let  $S \subset \mathbb{R}^d$  be a bounded set,  $|S| > 0$ ,  $|\partial S| = 0$ , and  $\aleph_n = \{X_1, \dots, X_n\}$  iid random vectors with uniform distribution on  $S$ . Then, there exists  $a_-^S = a_-^S(w, n)$ ,  $a_+^S = a_+^S(w, n)$  such that  $a_-^S \rightarrow \alpha_A$  and  $a_+^S \rightarrow \alpha_A$  as  $w \rightarrow \infty$  and  $w/n \rightarrow 0$ , such that if  $\gamma = \frac{n}{|S|} w^{d-1} e^{-w}$ ,*

$$\exp(-\gamma a_+^S |S|) \leq \mathbb{P}(nV(\aleph_n) < w) \leq \exp(-\gamma a_-^S |S|). \quad (8)$$

*The functions  $a_+^S$  and  $a_-^S$  only depend on the “shape” of  $S$  (i.e. are invariant by similarity transformations). Without loss of generality they can be chosen such that, for all  $w' \geq w$  and  $n' \geq n$ :  $a_+^S(w', n') \leq a_+(w, n)$  and  $a_-^S(w', n') \geq a_-(w, n)$ .*

Next we settle two lemmas whose proofs are quite similar. The first one (Lemma 2) gives a first rough upper bound for the maximal spacing. The second one (Lemma 3) bounds a sort “constrained” maximal-spacing (the centre  $x$  of the largest set  $x + rA$  missing the observation is constrained to be in a “small” subset of the support).

Recall first (see [1]) that, since  $A$  is convex, there exist  $\varepsilon_0 > 0$  such that,

$$\text{for all } \varepsilon \leq \varepsilon_0, A^{-\varepsilon} \neq \emptyset \text{ and } |A^{-\varepsilon}| = |A| - \varepsilon |\partial A|_{d-1} + o(\varepsilon). \quad (9)$$

It can also be proved easily that,

$$\text{for all } r > 0, \text{ and for all } x \in \mathcal{B}(0, \varepsilon_0/r), \ x + (rA)^{-\|x\|} \subset rA. \quad (10)$$

**Lemma 2.** *Let  $\aleph_n = \{X_1, \dots, X_n\}$  be iid random vectors in  $\mathbb{R}^d$ , with common density  $f$ . Assume that  $f$  has bounded support  $S$  and there exist  $0 < f_0 < f_1 < \infty$  such that  $f_0 \leq f(x) \leq f_1$  for all  $x \in S$ . Then, for all  $r_f > 0$  such that  $r_f^d > 2f_1/f_0$  we have,*

$$\Delta(\aleph_n) \leq r_f \left( \frac{\log(n)}{n} \right)^{1/d} \text{ eventually almost surely.}$$

*Proof.* First observe that  $S$  can be covered with  $N(S, n^{-1/d}) \leq C_S n$  balls of radius  $n^{-1/d}$  centered at some points  $\{x_1, \dots, x_{\nu_n}\} \subset S$ . Denote  $w_n = r_f \left( \frac{\log(n)}{n} \right)^{1/d}$  with  $r_f^d > 2f_1/f_0$ . First observe that  $\Delta(\aleph_n) \geq w_n \Leftrightarrow \exists x \in S$ , such that  $x + w_n f(x)^{-1/d} A \subset S \setminus \aleph_n$ , then  $\Delta(\aleph_n) \geq w_n \Rightarrow \exists x \in S$ , such that  $x + w_n f_1^{-1/d} A \subset S \setminus \aleph_n$ . Applying (10), for sufficiently large  $n$  (that is possible because  $n^{-1/d} \ll w_n$ ) we get

$$\Delta(\aleph_n) \geq w_n \Rightarrow \exists x_i \text{ such that } x_i + (w_n f_1^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n. \quad (11)$$

Next notice that,

$$\begin{aligned} \mathbb{P}\left(x_i + (w_n f_1^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n\right) &= \left(1 - \mathbb{P}_X\left(x_i + (w_n f_1^{-1/d} A)^{-1/n^{1/d}}\right)\right)^n \\ &\leq \left(1 - f_0 \left|(w_n f_1^{-1/d} A)^{-1/n^{1/d}}\right|\right)^n \\ &\leq \left(1 - \left(\frac{f_0}{f_1} w_n^d - \frac{f_0}{f_1^{\frac{d-1}{d}}} w_n^{d-1} n^{-1/d} (1 + o(1))\right)\right)^n. \end{aligned}$$

The last inequality is obtained using (9). Since  $w_n \gg n^{-1/d}$ , we finally get

$$\mathbb{P}\left(x_i + (w_n f_1^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n\right) \leq \left(1 - \frac{f_0}{f_1} w_n^d (1 + o(1))\right)^n.$$

From this inequality and (11) it follows that,

$$\begin{aligned} \mathbb{P}\left(\Delta(\aleph_n) \geq r_f (\log(n)/n)^{1/d}\right) &\leq N(S, n^{-1/d}) \left(1 - \frac{f_0}{f_1} w_n^d (1 + o(1))\right)^n \\ &\leq C_S n \exp\left(-\frac{f_0}{f_1} n w_n^d (1 + o(1))\right), \end{aligned}$$

and therefore,

$$\mathbb{P}\left(\Delta(\aleph_n) \geq r_f (\log(n)/n)^{1/d}\right) \leq C_S n^{1-r_f^d f_0/f_1+o(1)}.$$

Finally, since  $r_f^d > 2f_1/f_0$  we have  $\sum \mathbb{P}(\Delta(\aleph_n) \geq r_f(\log(n)/n)^{1/d}) < \infty$ . Thus, the Borel-Cantelli Lemma entails that  $\Delta(\aleph_n) \leq r_f(\log(n)/n)^{1/d}$  eventually almost surely.  $\square$

**Lemma 3.** *Let  $\aleph_n = \{X_1, \dots, X_n\}$  be iid random vectors in  $\mathbb{R}^d$  with common distribution  $\mathbb{P}_X$  supported on a compact set  $S$  and density  $f$  continuous on  $S$ . Assume that there exists  $f_0 > 0$  such that  $f(x) \geq f_0 \forall x \in S$ . Let  $G_n$  be a sequence of sets included in  $S$ , with the following property: there exist  $C$  such that  $N(G_n, n^{-1/d}) \leq Cn^{1-a}(\log(n))^b$  for some  $a > 0$  and  $b > 0$ . Let  $A$  be a compact and convex set with  $|A| = 1$  such that its barycenter is the origin of  $\mathbb{R}^d$ . Let us denote*

$$\begin{aligned}\Delta(\aleph_n, G_n) &= \sup \left\{ r : \exists x \in G_n \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset S \setminus \aleph_n \right\}, \\ V(\aleph_n, G_n) &= \Delta^d(\aleph_n, G_n), \\ U(\aleph_n, G_n) &= nV(\aleph_n, G_n) - \log(n) - (d-1)\log(\log(n)) - \log(\alpha_A).\end{aligned}$$

Then,  $\mathbb{P}(U(\aleph_n, G_n) \geq -\log(\log(n))) \rightarrow 0$ .

*Proof.* Let us first cover  $G_n$  with  $\nu_n = N(G_n, n^{-1/d})$  balls of radius  $n^{-1/d}$ , centred at some points  $\{x_1, \dots, x_{\nu_n}\}$  belonging to  $S$ , and choose  $w_n = (\frac{\log(n) + (d-2)\log(\log(n)) + \log(\alpha_A)}{n})^{1/d}$  (observe that  $w_n \gg (1/n)^{1/d}$ ). As in the proof of Lemma 2 we have,

$$\Delta(\aleph_n) \geq w_n \Leftrightarrow \exists x \in G_n, \text{ such that } x + w_n f(x)^{-1/d} A \subset S \setminus \aleph_n.$$

which implies,

$$\Delta(\aleph_n) \geq w_n \Rightarrow \exists x_i \exists x \in \mathcal{B}(x_i, n^{-1/d}), \text{ such that } x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n.$$

Therefore,

$$\mathbb{P}\left(x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n\right) = \left(1 - \mathbb{P}_X\left(x_i + (w_n f(x)^{-1/d} A)^{-(1/n^{1/d})}\right)\right)^n.$$

With rough bounds on the density,

$$\mathbb{P}_X\left(x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}}\right) \geq \frac{\min_{t \in S \cap (x_i + w_n f_1^{-1/d} A)} f(t)}{\max_{t \in S \cap \mathcal{B}(x_i, n^{-1/d})} f(t)} w_n^d (1 + o(1)).$$

Since  $f$  is uniformly continuous on  $S$ ,  $A$  is bounded,  $w_n \rightarrow 0$  and  $n^{-1/d} \rightarrow 0$ , for all  $c < 1$  there exist  $n_c$  such that for all  $n \geq n_c$

$$\mathbb{P}\left(x_i + (w_n f(x)^{-1/d} A)^{-1/n^{1/d}} \subset S \setminus \aleph_n\right) \leq \left(1 - c w_n^d (1 + o(1))\right)^n,$$

then,

$$\mathbb{P}(\Delta(\aleph_n, G_n) \geq w_n) \leq C n^{1-a} (\log n)^b (1 - c w_n^d (1 + o(1)))^n.$$

Taking  $c = 1 - a/2$ , we finally get that

$$\mathbb{P}(U(\aleph_n, G_n) \geq -\log(\log(n))) \leq C\alpha_A^{-1+a/2} n^{-a/2} (\log(n))^{b-(1-a/2)(d-2)} (1 + o(1)) \rightarrow 0.$$

□

The next lemma relates the behaviour of the maximal-spacing for two different densities having the same support.

**Lemma 4.** *Let us consider  $f$  and  $h$ , two densities with compact support  $S$  such that,  $h(x) > h_0$  for all  $x \in S$  and  $\max_{x \in S} |f(x) - h(x)| \leq \varepsilon h_0$  for a given  $\varepsilon \in (0, 1/2)$ . Denote by  $n_0 = \lfloor n(1 - 2\varepsilon) \rfloor$  and  $n_1 = \lceil n(1 + 2\varepsilon) \rceil$  the floor and ceiling of  $n(1 - 2\varepsilon)$  and  $n(1 + 2\varepsilon)$  respectively. For any  $w \in \mathbb{R}$ , let us define  $w_{n,0} = \frac{w(1-2\varepsilon-n^{-1})}{(1+\varepsilon)}$  and  $w_{n,1} = \frac{w(1-\varepsilon)}{1+2\varepsilon}$ . Then,*

$$\mathbb{P}(n_0 V(\mathcal{Y}_{n_0}) \leq w_{n,0}) \left(1 - \frac{1-\varepsilon}{n\varepsilon}\right) \leq \mathbb{P}(nV(\aleph_n) \leq w), \quad (12)$$

and

$$\mathbb{P}(nV(\aleph_n) \leq w) \leq \mathbb{P}(n_1 V(\mathcal{Y}_{n_1}) \leq w_{n,1}) \left(1 - \frac{1+2\varepsilon+n^{-1}}{(n\varepsilon+1)(1+\varepsilon)}\right)^{-1}, \quad (13)$$

where  $\mathcal{Y}_{n_0} = \{Y_1, \dots, Y_{n_0}\}$  and  $\mathcal{Y}_{n_1} = \{Y_1, \dots, Y_{n_1}\}$  are iid random vectors on  $\mathbb{R}^d$ , with density  $h$ , and  $\aleph_n = \{X_1, \dots, X_n\}$  are iid random vectors with density  $f$ .

*Proof.* We first prove (12). Observe that  $X$  can be generated from the following mixture: with probability  $p = 1 - \varepsilon$ ,  $X$  is drawn with density  $h$ , and, with probability  $1 - p$ ,  $X$  is drawn with the law given by the density  $g(x) = \frac{f(x) - h(x)(1-\varepsilon)}{\varepsilon} \mathbb{I}_S(x)$ . Let us denote by  $N_0$  the number of points drawn according to  $h$  on  $S$  and  $\aleph_{N_0}^* = \{Y_1, \dots, Y_{N_0}\}$  the associated sample. Let us recall that

$$\Delta(\aleph_n) = \sup \left\{ r : \exists x \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset S \setminus \aleph_n \right\}.$$

Observe that

$$\sup_x h_0 |f(x)/h(x) - 1| \leq \sup_x h(x) |f(x)/h(x) - 1| \leq h_0 \varepsilon,$$

so  $f(x)/h(x) \leq 1 + \varepsilon$ . Then we have,

$$\Delta(\aleph_n) \leq (1 + \varepsilon)^{1/d} \sup \left\{ r : \exists x \text{ such that } x + \frac{r}{h(x)^{1/d}} A \subset S \setminus \aleph_n \right\}.$$

From the inclusion  $\aleph_{N_0}^* \subset \aleph_n$  we get

$$\Delta(\aleph_n) \leq (1 + \varepsilon)^{1/d} \sup \left\{ r : \exists x \text{ such that } x + \frac{r}{h(x)^{1/d}} A \subset S \setminus \aleph_{N_0}^* \right\},$$

and therefore  $\Delta(\aleph_n) \leq (1 + \varepsilon)^{1/d} \Delta(\aleph_{N_0}^*)$ , which entails that  $V(\aleph_n) \leq (1 + \varepsilon) V(\aleph_{N_0}^*)$ . Then, for all  $w > 0$ ,

$$\mathbb{P}(nV(\aleph_n) \leq w) \geq \mathbb{P}((1 + \varepsilon)nV(\aleph_{N_0}^*) \leq w),$$

and

$$\mathbb{P}(nV(\aleph_n) \leq w) \geq \mathbb{P}\left(\left((1+\varepsilon)nV(\aleph_{N_0}^*) \leq w\right) \cap (N_0 \geq n_0)\right).$$

For  $N_0 \geq n_0$ , let us denote by  $\mathcal{Y}_{n_0} = \{Y_1, \dots, Y_{n_0}\}$  the  $n_0$  first values of  $\aleph_{N_0}^*$ . Clearly we have  $V(\aleph_{N_0}^*) \leq V(\mathcal{Y}_{n_0})$  so,

$$\mathbb{P}^{N_0 \geq n_0} \left( (1+\varepsilon)nV(\aleph_{N_0}^*) \leq w \right) \geq \mathbb{P}\left( (1+\varepsilon)nV(\mathcal{Y}_{n_0}) \leq w \right),$$

where  $\mathbb{P}^{N_0 \geq n_0}$  denotes the conditional probability given  $N_0 \geq n_0$ . Therefore,

$$\mathbb{P}(nV(\aleph_n) \leq w) \geq \mathbb{P}\left(n_0V(\mathcal{Y}_{n_0}) \leq \frac{wn_0}{(1+\varepsilon)n}\right) \mathbb{P}(N_0 \geq n_0).$$

On the other hand, since  $N_0 \sim \text{Bin}(n, 1-\varepsilon)$ , we obtain,

$$\mathbb{P}(N_0 < n_0) = \mathbb{P}\left(N_0 - (1-\varepsilon)n < n_0 - (1-\varepsilon)n\right) \leq \mathbb{P}(N_0 - (1-\varepsilon)n \leq -\varepsilon n).$$

Since  $n_0 = \lfloor n(1-2\varepsilon) \rfloor$ ,  $n_0 - n(1-\varepsilon) \leq -\varepsilon n$ , which together with Chebyshev's inequality entails that

$$\mathbb{P}(N_0 < n_0) \leq \frac{n\varepsilon(1-\varepsilon)}{n^2\varepsilon^2} = \frac{(1-\varepsilon)}{n\varepsilon},$$

and then,

$$\mathbb{P}(N_0 \geq n_0) \geq 1 - \frac{(1-\varepsilon)}{n\varepsilon}.$$

Let us denote by  $w_{n,0} = \frac{w(1-2\varepsilon-n^{-1})}{(1+\varepsilon)}$ . Since  $n(1-2\varepsilon) - 1 \leq n_0$  we have  $w_{n,0} \leq \frac{wn_0}{(1+\varepsilon)n}$ , from where it follows that

$$\mathbb{P}(nV(\aleph_n) \leq w) \geq \mathbb{P}\left(n_0V(\mathcal{Y}_{n_0}) \leq w_{n,0}\right) \left(1 - \frac{1-\varepsilon}{n\varepsilon}\right).$$

Equation (13) is proved in the same way. We just provide a sketch of the proof. The key point for the proof of (12) was to think the law of a random variable  $Y$  drawn with the density  $h$  as the following mixture: with probability  $p = \frac{1}{1+\varepsilon}$ ,  $Y$  as a random variable with density  $f$ , and, with probability  $1-p$ ,  $Y$  is drawn with density  $g(x) = \frac{h(x)(1+\varepsilon)-f(x)}{\varepsilon} \mathbb{I}_S(x)$ . Next, we consider a sample  $\mathcal{Y}_{n_1} = \{Y_1, \dots, Y_{n_1}\}$  of iid copies of  $Y$ , (that follows a law given by  $h$ ). Denote by  $N$  the number of the points that drawn according to the density  $f$  and  $\mathcal{Y}_N^* = \{X_1, \dots, X_N\}$  these points. The rest of the proof follows using the same argument to prove (12).  $\square$

## 4.2 Uniform mixture on disjoint supports

**Proposition 1.** *Let  $E_1, \dots, E_k$  be subsets of  $\mathbb{R}^d$  such that for all  $i \neq j \Rightarrow \overline{E_i} \cap \overline{E_j} = \emptyset$ , and  $0 < |E_i| < \infty$  for all  $i$ . Let  $\aleph_n = \{X_1, \dots, X_n\}$  be iid random vectors in  $S = \cup_i E_i$  with density*

$$f(x) = \sum_{i=1}^k p_i \mathbb{I}_{E_i}(x),$$

where  $p_1, \dots, p_k$  are positive real numbers. Then,

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \text{when } n \rightarrow \infty.$$

*Proof.* First let us introduce some notation, for  $i = 1, \dots, k$

- $N_i = \#\{\aleph_n \cap E_i\}$  denotes the number of data points in  $E_i$ . Notice that  $N_i \sim \text{Bin}(n, p_i | E_i|)$ .
- $\aleph_{N_i}^i = \{X_{i_1}, \dots, X_{i_{N_i}}\}$  denotes the subsample of  $\aleph_n$  that falls in  $E_i$ . Observe that they all are uniformly distributed.
- $a_i = p_i |E_i|$  for  $i = 1, \dots, k$ , that fulfils  $\sum a_i = 1$ ,  $a_0 = \min_i a_i$ ,  $A_0 = \max_i a_i$  and  $C = \sum \frac{1-a_i}{a_i}$ .
- $\varepsilon_{n,i} = \frac{N_i - a_i n}{n a_i}$ .

Since the support of  $f$  is  $\cup_i \overline{E_i}$ , and by assumption  $i \neq j \Rightarrow \overline{E_i} \cap \overline{E_j} = \emptyset$ , we have

$$\Delta(\aleph_n) = \sup \left\{ r : \exists x \exists i \text{ such that } x + \frac{r}{p_i^{1/d}} A \subset E_i \setminus \aleph_n \right\},$$

so

$$\Delta(\aleph_n) = \max_i \sup \left\{ r : \exists x \text{ such that } x + \frac{r |E_i|^{1/d}}{(|E_i| p_i)^{1/d}} A \subset E_i \setminus \aleph_{N_i}^i \right\}, \quad (14)$$

while

$$\Delta(\aleph_{N_i}^i) = \sup \left\{ r' : \exists x \in E_i \text{ such that } x + r' |E_i|^{1/d} A \subset E_i \setminus \aleph_{N_i}^i \right\}. \quad (15)$$

From (14) and (15) we derive that

$$\Delta(\aleph_n) = \max_i \left\{ (|E_i| p_i)^{1/d} \Delta(\aleph_{N_i}^i) \right\} \text{ and } V(\aleph_n) = \max_i \left\{ |E_i| p_i V(\aleph_{N_i}^i) \right\},$$

which entails that

$$\mathbb{P}(nV(\aleph_n) \leq w) = \prod_i \mathbb{P} \left( N_i V(\aleph_{N_i}^i) \leq \frac{w N_i}{a_i n} \right).$$

Let  $\mathbb{P}^{\vec{n}}(A) = \mathbb{P}(A | N_1 = n_1, \dots, N_k = n_k)$  stand for the conditional probability given the number of points that fall in each  $E_i$ . We have that,

$$\mathbb{P}^{\vec{n}}(nV(\aleph_n) \leq w) = \prod_{i=1}^k \mathbb{P}^{\vec{n}}(n |E_i| p_i V(\aleph_{n_i}^i) \leq w) = \prod_{i=1}^k \mathbb{P}^{\vec{n}} \left( n_i V(\aleph_{n_i}^i) \leq \frac{w n_i}{n |E_i| p_i} \right).$$

Now, taking  $w_{n,i} = \frac{w n_i}{n |E_i| p_i}$ ,  $\gamma_{n,i} = \frac{n_i w_{n,i}^{d-1} e^{-w_{n,i}}}{|E_i|}$  and applying Lemma 1 we obtain,

$$\exp \left( - \sum_{i=1}^k \gamma_{n,i} a_+^{E_i} |E_i| \right) \leq \mathbb{P}^{\vec{n}}(nV(\aleph_n) \leq w) \leq \exp \left( - \sum_{i=1}^k \gamma_{n,i} a_-^{E_i} |E_i| \right).$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^k \gamma_{n,i} a_+^{E_i} |E_i| &= \sum_{i=1}^k n_i \left( \frac{wn_i}{n|E_i|p_i} \right)^{d-1} \exp\left(-\frac{wn_i}{n|E_i|p_i}\right) a_+^{E_i} \\ &= \sum_{i=1}^k n_i w^{d-1} (1 + \varepsilon_i)^{d-1} \exp(-w(1 + \varepsilon_i)) a_+^{E_i}(w_{n,i}, n_i).\end{aligned}$$

Let  $\varepsilon_n = \max_i |\varepsilon_{n,i}|$  and  $\varepsilon_{a_+} = \max_i \frac{|a_+^{E_i}(w_{n,i}, n_i) - \alpha_A|}{\alpha_A}$ , then we have

$$\sum_{i=1}^k \gamma_{n,i} a_+^{E_i} |E_i| \leq n w^{d-1} \exp(-w) \alpha_A (1 + \varepsilon_n)^{d-1} \exp(w\varepsilon_n) (1 + \varepsilon_{a_+}).$$

Taking  $w = w_n = x + \log(n) + (d-1)\log(\log(n)) + \log(\alpha_A)$ , we obtain that  $nV \leq w \Leftrightarrow U \leq x$ , which implies that

$$\mathbb{P}^{\vec{n}}(U(\aleph_n) \leq x) \geq \exp\left(-e^{-x}(1 + \varepsilon)^{d-1} \exp(\log(n)\varepsilon_n)(1 + \varepsilon_{a_+})(1 + o_n(1))\right).$$

In the same way it can be proved,

$$\mathbb{P}^{\vec{n}}(U(\aleph_n) \leq x) \leq \exp\left(-e^{-x}(1 - \varepsilon_n)^{d-1} \exp(-\log(n)\varepsilon_n)(1 - \varepsilon_{a_-})(1 + o_n(1))\right).$$

where we denoted  $\varepsilon_{a_-} = \max_i \frac{|a_-^{E_i}(w_{n,i}, n_i) - \alpha_A|}{\alpha_A}$ .

Suppose that  $\varepsilon_n = \max_i |\varepsilon_{n,i}| \leq 1/\log(n)^2$ , then, if  $n \geq 5$ ,  $a_0 n/2 \leq N_i \leq n$  for all  $i$ , which imply that for all  $i$ ,  $w_{n,i} \geq \log(n)a_0/(2A_0) \rightarrow \infty$  and  $w_{n,i}/n \leq (x + \log(n) + (d-1)\log(\log(n)) + \log(\alpha_A))/(na_0) \rightarrow 0$ . Then  $\varepsilon_{a_-}$  and  $\varepsilon_{a_+}$  converges to 0, according to Lemma 4. Therefore

$$\mathbb{P}^{\varepsilon_n \leq 1/\log(n)^2}(U(\aleph_n) \leq x) \rightarrow \exp(-\exp(-x)) \quad \text{when } n \rightarrow \infty. \quad (16)$$

Since

$$\mathbb{P}\left(\max_i |\varepsilon_{n,i}| \geq \frac{1}{\log(n)^2}\right) = \mathbb{P}\left(\bigcup_i |\varepsilon_{n,i}| \geq \frac{1}{\log(n)^2}\right) \leq \sum_{i=1}^k \mathbb{P}\left(|\varepsilon_{n,i}| \geq \frac{1}{\log(n)^2}\right),$$

from Chebyshev's inequality we obtain

$$\mathbb{P}\left(|\varepsilon_{n,i}| \geq \frac{1}{\log(n)^2}\right) \leq \log(n)^4 \mathbb{V}(\varepsilon_{n,i}^2) \quad \text{where} \quad \mathbb{V}(\varepsilon_{n,i}^2) = \frac{1 - a_i}{na_i},$$

and therefore

$$\mathbb{P}\left(\varepsilon_n \geq \frac{1}{\log(n)^2}\right) \leq C \frac{(\log(n))^4}{n}. \quad (17)$$

Finally, from equations (16) and (17) we get,

$$\mathbb{P}(U(\aleph_n) \leq x) \rightarrow \exp(-\exp(-x)) \quad \text{when } n \rightarrow \infty,$$

which concludes the proof.  $\square$



### 4.3 Uniform mixture

**Proposition 2.** *Let  $E_1, \dots, E_k$  be subsets of  $\mathbb{R}^d$  such that,*

- 1)  $i \neq j \Rightarrow |\overline{E_i} \cap \overline{E_j}| = 0$ .
- 2)  $0 < |E_i| < \infty$  for  $i = 1, \dots, k$ .
- 3) *There exists a constant  $C > 0$  such that, for all  $i$ , for all  $\varepsilon > 0$ ,  $N(\partial E_i, \varepsilon) \leq C\varepsilon^{-d+1}$ .*

Let  $\aleph_n = \{X_1, \dots, X_n\}$  be iid random vectors with density

$$f(x) = \sum_{i=1}^k p_i \mathbb{I}_{\dot{E}_i},$$

where  $p_1, \dots, p_k$  are positive real numbers. If there exist constants  $r_0 > 0$  and  $c > 1 - 1/d$  such that, for all  $r \leq r_0$  and all  $x \in \cup_i \dot{E}_i$ ,

$$\frac{\min_{t \in S \cap B(x, r)} f(t)}{\max_{t \in S \cap B(x, r)} f(t)} \geq c,$$

then,

$$U(\aleph_n) \xrightarrow{\mathcal{L}} U \quad \text{when } n \rightarrow \infty.$$

*Proof.* We start by introducing some definitions and notation.

$$\begin{aligned} \dot{\Delta}(\aleph_n) &= \sup \left\{ r : \exists x \exists i, \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset \dot{E}_i \setminus \aleph_n \right\}, \\ \dot{V}(\aleph_n) &= \dot{\Delta}^d(\aleph_n), \\ \dot{U}(\aleph_n) &= n \dot{V}(\aleph_n) - \log(n) - (d-1) \log(\log(n)) - \log(\alpha_A) \end{aligned}$$

With the same ideas used to prove Proposition 1 (and the fact that  $|E_i| = |\dot{E}_i|$ ) it follows that  $\dot{U}(\aleph_n) \xrightarrow{\mathcal{L}} U$ . Let  $F_n(x) = \mathbb{P}(\dot{U}(\aleph_n) \leq x)$ . Clearly  $U(\aleph_n) \geq \dot{U}(\aleph_n)$ , and therefore

$$\mathbb{P}(U(\aleph_n) \leq x) \leq F_n(x) \rightarrow \exp(-\exp(-x)). \quad (18)$$

In order to prove the other inequality let us define

- $G = \bigcup_{i,j} (\overline{E_i} \cap \overline{E_j})$ .
- $p_0 = \min_i p_i$ .
- $\rho_A = \max_{x \in A} \|x\|$ .
- $\rho_n = (r_f \rho_A / p_0^{1/d}) (\log(n)/n)^{1/d}$  with  $r_f$  such that  $\Delta(\aleph_n) \leq r_f (\log(n)/n)^{1/d}$  eventually almost surely (whose existence follows from Lemma 2). Notice that condition 3) ensures that  $N(G^{\rho_n}, n^{-1/d}) = \mathcal{O}(n^{1-1/d} (\log(n))^{1/d})$ .

- $\Delta(\aleph_n, S \setminus G^{\rho_n}) = \sup \left\{ r : \exists x \in S \setminus G^{\rho_n} \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset S \setminus \aleph_n \right\}.$
- $\Delta(\aleph_n, G^{\rho_n}) = \sup \left\{ r : \exists x \in G^{\rho_n} \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset S \setminus \aleph_n \right\}.$

Clearly we have that,

$$\Delta(\aleph_n) = \max \left\{ \Delta(\aleph_n, S \setminus G^{\rho_n}), \Delta(\aleph_n, G^{\rho_n}) \right\}. \quad (19)$$

For the chosen  $\rho_n$ , we are going to prove that,

$$\dot{\Delta}(\aleph_n) \geq \Delta(\aleph_n, S \setminus G^{\rho_n}) \text{ eventually almost surely.} \quad (20)$$

Let us suppose first that  $\Delta(\aleph_n, S \setminus G^{\rho_n}) \leq r_f(\log(n)/n)^{1/d}$  (which holds, e.a.s., due to Lemma 2) then

$$\text{for all } \varepsilon > 0 \text{ there exists } x_\varepsilon \in S \setminus G^{\rho_n} \text{ such that } x_\varepsilon + \frac{\Delta(\aleph_n, S \setminus G^{\rho_n}) - \varepsilon}{f(x_\varepsilon)^{1/d}} A \subset S \setminus \aleph_n,$$

and

$$x_\varepsilon + \frac{\Delta(\aleph_n, S \setminus G^{\rho_n}) - \varepsilon}{f(x_\varepsilon)^{1/d}} A \subset \mathcal{B}\left(x_\varepsilon, \rho_A \frac{r_f(\log(n)/n)^{1/d} - \varepsilon}{p_0^{1/d}}\right) \subset \mathcal{B}(x_\varepsilon, \rho_n).$$

From  $d(x_\varepsilon, G) \geq \rho_n$  we get  $x_\varepsilon + \frac{\Delta(\aleph_n, S \setminus G^{\rho_n}) - \varepsilon}{f(x_\varepsilon)^{1/d}} A \subset \bigcup_i \dot{E}_i \setminus \aleph_n$ . Then, for all  $\varepsilon > 0$ ,  $\dot{\Delta}(\aleph_n) \geq \Delta(\aleph_n, S \setminus G^{\rho_n}) - \varepsilon$  e.a.s., which concludes the proof of (20).

By (19), introducing  $U(\aleph_n, G^{\rho_n}) = n\Delta(\aleph_n, G^{\rho_n})^d - \log(n) - (d-1)\log(\log(n)) - \log(\alpha_A)$ , one can bound  $\mathbb{P}(U(\aleph_n) \geq x)$  for all  $x$ , as follows:

$$\begin{cases} \mathbb{P}(U(\aleph_n) \geq x) \leq \mathbb{P}(\dot{U}(\aleph_n) \geq x) + \mathbb{P}(U(\aleph_n, G^{\rho_n}) \geq -\log(\log(n))) & \text{if } x \geq -\log(\log(n)) \\ \mathbb{P}(U(\aleph_n) \geq x) \leq 1 & \text{if } x \leq -\log(\log(n)) \end{cases}$$

By Lemma 3 we obtain:

$$\begin{cases} \mathbb{P}(U(\aleph_n) \leq x) \geq F_n(x) + o(1) & \text{if } x \geq -\log(\log(n)) \\ \mathbb{P}(U(\aleph_n) \leq x) \geq 0 & \text{if } x \leq -\log(\log(n)). \end{cases} \quad (21)$$

Finally using (18), (21) and that  $\mathbb{P}(U \leq -\log(\log(n))) = 1/n \rightarrow 0$  we conclude the proof.  $\square$

## Proof of Theorem 2

Let  $c_n = (\log(n)/n)^{\frac{1}{3d}}$ . Take a “mesh” of  $\mathbb{R}^d$  with small squares of side  $c_n$ ,

$$\prod_{i=1}^d [k_i c_n, (k_i + 1)c_n] \quad \text{with } k_i \in \mathbb{N},$$

and denote by  $m_n \leq |S|c_n^{-d}$  the number of these squares  $\{C_1, \dots, C_{m_n}\}$  that are included in  $S$ .

Like in the proof of Proposition 2 let us denote,

$$\begin{aligned}\mathring{\Delta}(\mathfrak{N}_n) &= \sup \left\{ r : \exists x \exists i, \text{ such that } x + \frac{r}{f(x)^{1/d}} A \subset \mathring{C}_i \setminus \mathfrak{N}_n \right\}, \\ \mathring{V}(\mathfrak{N}_n) &= \mathring{\Delta}^d(\mathfrak{N}_n), \\ \mathring{U}(\mathfrak{N}_n) &= n\mathring{V}(\mathfrak{N}_n) - \log(n) - (d-1) \log(\log(n)) - \log(\alpha_A).\end{aligned}$$

From the inclusion  $\bigcup_{i=1}^{m_n} \mathring{C}_i \subset S$  it follows that,  $\mathbb{P}(U(\mathfrak{N}_n) \leq x) \leq \mathbb{P}(\mathring{U}(\mathfrak{N}_n) \leq x)$ .

Like in the proof of Proposition 1 let us denote, for  $i = 1, \dots, m_n$ .

- $N_i = \#\{\mathfrak{N}_n \cap C_i\}$ ,
- $a_i = \int_{C_i} f(t)dt$ ;  $a_0 = \min_i a_i$ ;  $A_0 = \max_i a_i$  and  $C = \sum \frac{1-a_i}{a_i}$ . Observe that  $\sum a_i = 1$  and  $a_0 \geq f_0 c_n^d$ .
- $\mathfrak{N}_{N_i}^i = \{X_{i_1}, \dots, X_{i_{N_i}}\}$ , the subsample of  $\mathfrak{N}_n$  that belongs to  $C_i$ . Observe that  $X_{i_j}$  for  $j = 1, \dots, N_i$  has density  $f_i(x) = (f(x)/a_i)\mathbb{I}_{C_i}(x)$ .
- $\varepsilon_{n,i} = \frac{N_i - a_i n}{na_i}$ ,  $\varepsilon_n = \max_i \varepsilon_i$ .

We start with some asymptotic properties about  $\varepsilon_{n,i}$  and  $\varepsilon_n$ . If we bound  $|a_i| \geq f_0 |c_n|^d$  and apply Hoeffding's inequality we get

$$\mathbb{P}(\log(n)|\varepsilon_{n,i}| \geq t) \leq 2 \exp \left( -2t^2 f_0^2 (\log(n))^{-4/3} n^{1/3} \right),$$

and,

$$\mathbb{P}(\log(n)|\varepsilon_n| \geq t) \leq \frac{2|S|n^{1/3}}{(\log n)^{1/3}} \exp \left( -2t^2 f_0^2 (\log(n))^{-4/3} n^{1/3} \right).$$

Borel-Cantelli Lemma entails  $(\log(n))\varepsilon_n \xrightarrow{a.s.} 0$ . Then, with probability 1, for  $n$  large enough,

$$\frac{f_0}{2} (\log n)^{1/3} n^{2/3} \leq N_i \leq 2f_1 (\log n)^{1/3} n^{2/3} \quad \text{for } i = 1, \dots, m_n. \quad (22)$$

In what follows  $n$  is large enough so that (22) is fulfilled.

Proceeding exactly as in the proof of Proposition 1 we can derive that

$$\mathring{\Delta}(\mathfrak{N}_n) = \max_i \sup \left\{ r : \exists x \text{ such that } x + \frac{ra_i^{1/d}}{(a_i f_i(x))^{1/d}} A \subset \mathring{C}_i \setminus \mathfrak{N}_{N_i}^i \right\},$$

and therefore

$$\mathring{\Delta}(\mathfrak{N}_n) = \max_i \left\{ a_i^{1/d} \mathring{\Delta}(\mathfrak{N}_{N_i}^i) \right\} \quad \text{and} \quad V(\mathfrak{N}_n) = \max_i \left\{ a_i V(\mathfrak{N}_{N_i}^i) \right\}.$$

First we to bound  $\mathbb{P}(U_n(\aleph_n) \geq x)$  from above. As in Proposition 1

$$\mathbb{P}(nV(\aleph_n) \leq w_n) \leq \mathbb{P}(n\hat{V}(\aleph_n) \leq w_n) = \prod_{i=1}^{m_n} \mathbb{P}\left(N_i \Delta^d(\aleph_{N_i}) \leq \frac{w_n N_i}{a_i n}\right). \quad (23)$$

At any of the small squares  $C_i$ , by Hölder continuity, the density is close to the uniform density, that will allow us to apply Lemma 4 with  $h = 1/|C_i|\mathbb{I}_{C_i}$ . More precisely: for all  $i$  and for all  $y \in C_i$ ,

$$\left|f_i(y)|C_i| - 1\right| = \left|\frac{f(y)}{a_i}|C_i| - 1\right| = \frac{1}{a_i} \left|\int_{C_i} f(y)dt - \int_{C_i} f(t)dt\right| \leq \frac{1}{a_i} K_f \int_{C_i} |y - t|^\beta dt.$$

Since  $|y - t| \leq \sqrt{d}c_n$ , if we denote  $A_f = K_f f_0^{-1} d^{\beta/2}$  we derive that

$$\left|f_i(y)|C_i| - 1\right| \leq \frac{1}{a_i} K_f \sqrt{d}^\beta c_n^{d+\beta} \leq A_f c_n^\beta \quad \forall y \in C_i.$$

Let  $N'_i = \lceil N_i(1 + 2A_f c_n^\beta) \rceil$ ,  $w'_n = w_n \frac{1 - A_f c_n^\beta}{1 + 2A_f c_n^\beta}$  and  $\mathcal{Y}_{N'_i}$  a sample of  $N'_i$  variables uniformly drawn on  $C_i$ , then Lemma 4 implies that

$$\mathbb{P}\left(N_i \Delta^d(\aleph_{N_i}) \leq \frac{w_n N_i}{a_i n}\right) \leq \mathbb{P}\left(N'_i \Delta^d(\mathcal{Y}_{N'_i}) \leq \frac{w'_n N'_i}{a_i n}\right) \left(1 - \frac{1 + 2A_f c_n^\beta + N_i^{-1}}{(N_i A_f c_n^\beta)(1 + A_f c_n^\beta)}\right)^{-1}. \quad (24)$$

On the other hand, by (22), with probability one, for  $n$  large enough we have that,

$$\left(1 - \frac{1 + 2A_f c_n^\beta + N_i^{-1}}{(N_i A_f c_n^\beta)(1 + A_f c_n^\beta)}\right)^{-1} \leq \left(1 - \frac{2}{f_0 A_f} \frac{1}{(\log n)^{1/3+\beta/3d} n^{2/3-\beta/3d} (1 + o(1))}\right)^{-1} \quad \text{for all } i, \quad (25)$$

and,

$$\left(1 - \frac{1 + 2A_f c_n^\beta + N_i^{-1}}{(N_i A_f c_n^\beta)(1 + A_f c_n^\beta)}\right)^{-1} \geq \left(1 - \frac{1}{2f_1 A_f} \frac{1 + o(1)}{(\log n)^{1/3+\beta/3d} n^{2/3-\beta/3d}}\right)^{-1} \quad \text{for all } i. \quad (26)$$

Let us prove that,

$$\prod_{i=1}^{m_n} \left(1 - \frac{1 + 2A_f c_n^\beta + N_i^{-1}}{(N_i A_f c_n^\beta)(1 + A_f c_n^\beta)}\right)^{-1} \xrightarrow{a.s.} 1. \quad (27)$$

Since the right hand side of (25) can be express, for  $n$  large enough, as

$$\exp\left(-C(\log n)^{1/3+\beta/3d} n^{2/3-\beta/3d} (1 + o(1))\right),$$

being  $C$  a positive constant, we get, for  $n$  large enough,

$$\prod_{i=1}^{m_n} \left(1 - \frac{1 + 2A_f c_n^\beta + N_i^{-1}}{(N_i A_f c_n^\beta)(1 + A_f c_n^\beta)}\right)^{-1} \leq \exp\left(-C m_n (\log n)^{1/3+\beta/3d} n^{2/3-\beta/3d} (1 + o(1))\right) \rightarrow 1,$$

where the limit follows from  $|m_n(\log n)^{-1/3-\beta/3d}n^{-2/3+\beta/3d}| \leq (\log n)^{-\beta/3d}n^{-1/3+\beta/3d} \rightarrow 0$ . (27) is obtained doing the same with (26).

Now let us study the asymptotic behaviour of  $\prod_{i=1}^{m_n} \mathbb{P}\left(N'_i \Delta^d(\mathcal{Y}_{N'_i}) \leq \frac{w'_n N'_i}{a_i n}\right)$ . If we apply Lemma 1, (observe that the functions  $a_-^S$  only depends on the shape of  $S$ ) we get,

$$\prod_{i=1}^{m_n} \mathbb{P}\left(N'_i \Delta^d(\mathcal{Y}_{N'_i}) \leq \frac{w'_n N'_i}{a_i n}\right) \leq \exp\left(-\sum_{i=1}^{m_n} N'_i \left(\frac{N'_i \omega'_n}{a_i n}\right)^{d-1} \exp\left(-\frac{N'_i \omega'_n}{a_i n}\right) a_-^{[0,1]^d}\left(\frac{N'_i \omega'_n}{a_i n}, N'_i\right)\right).$$

Let,  $\varepsilon'_{n,i} = \frac{N_i}{a_i n} \frac{w'_n}{w_n} - 1$  for  $i = 1, \dots, m_n$ . Observe that  $\varepsilon'_{n,i} = (1 + \varepsilon_{n,i}) \frac{w'_n}{w_n} - 1 = (1 + \varepsilon_{n,i}) \frac{1 - A_f c_n^\beta}{1 + 2A_f c_n^\beta} - 1$  and  $\varepsilon'_n = \max_i |\varepsilon'_{n,i}|$ . Since  $(\log(n))\varepsilon_n \xrightarrow{a.s.} 0$ ,  $\varepsilon'_n$  fulfils  $\log(n)\varepsilon'_n \xrightarrow{a.s.} 0$  and for all  $i$ ,  $N'_i \geq N_i$ . The previous equation together with (23) entails,

$$\mathbb{P}(nV(\aleph_n) \leq w_n) \leq \exp\left(-n\omega_n^{d-1}(1-\varepsilon'_n)^{d-1} \exp(-w_n(1+\varepsilon'_n)) a_-^{[0,1]^d}(\min(w_n(1-\varepsilon'_n), \min_i N_i))\right). \quad (28)$$

If we choose  $w_n = x + n \log(n) + (d-1) \log(\log(n)) + \log(\alpha_A)$  in (28) we get,

$$\mathbb{P}(U(\aleph_n) \leq x) \leq \exp(-\exp(-x)) + o(1). \quad (29)$$

In order to conclude the proof of (2) we have to bound  $\mathbb{P}(U(\aleph_n) \leq x)$  from below. We provide just a sketch of the proof since the arguments are similar to those in Proposition 2, using Lemma 4 as in the proof of (29). Let us denote,

- $\rho_n = \frac{r_f \rho_A}{f_0^{1/d}} \left(\frac{\log(n)}{n}\right)^{1/d}$  with  $\rho_A = \max_{x \in A} \|x\|$ .
- $G_n = \cup_{i \neq j}^{m_n} (\overline{C_i} \cap \overline{C_j})$ .
- $H_n = S \setminus (\cup_i^{m_n} C_i)$ , notice that  $H_n \subset \partial S^{c_n}$ .

Proceeding as in Proposition 2 we have

$$U(\aleph_n) \leq \max \left\{ \mathring{U}(\aleph_n), U(\aleph_n, G_n^{\rho_n}), U(\aleph_n, H_n) \right\} \text{ eventually almost surely,}$$

and

$$\mathbb{P}(\mathring{U}(\aleph_n) \leq x) \geq \exp(-\exp(-x)) + o(1).$$

Then, reasoning as in Proposition 2 we get

$$\mathbb{P}(U(\aleph_n) \leq x) \geq \exp(-\exp(-x)) + o(1).$$

Finally, in order to conclude the proof of (2) it suffices to prove that  $G_n^{\rho_n}$  and  $H_n$  satisfies the hypothesis of Lemma 3.

$G_n$  is the union of less than  $m_n 2^d (d-1)$ -dimensional cubes of size  $c_n$ . Each of them can be cover by less than  $a_1 c_n^{d-1} \rho_n^{-d+1}$  balls of radius  $\rho_n$  (with  $a_1$  a positive

constant), centered at some points  $\{x_j^i\}_{i,j}$ . Since  $G_n^{\rho_n} \subset \bigcup_{i,j} \mathcal{B}(x_j^i, 2\rho_n)$ , and every  $\mathcal{B}(x_j^i, 2\rho_n)$  can be covered by  $a_2\rho_n^d n$  balls of radius  $n^{-1/d}$ ,  $G_n^{\rho_n}$  can be covered by less than  $\nu_n = m_n a_1 c_n^{d-1} \rho_n^{-d+1} a_2 \rho_n^d n = \mathcal{O}(n^{1-\frac{2}{3d}} (\log n)^{\frac{2}{3d}})$  balls of radius  $n^{-1/d}$ .

In the same way it can be proved that  $H_n$  can be covered with  $\mathcal{O}(n^{1-\frac{d+\kappa}{3d}} \log(n)^{\frac{d+\kappa}{3d}})$  balls of radius  $n^{-1/d}$ . Indeed, cover  $\partial S$  with  $\mathcal{O}(c_n^{-\kappa})$  balls of radius  $c_n$ , apply triangular inequality to obtain that the union of the balls with the same centre but with a radius  $c_n \sqrt{d}$  covers  $\partial S^{\sqrt{d}c_n}$  and then covers  $H_n$ , finally cover every of these balls by  $\mathcal{O}((c_n n^{1/d})^{-d})$  balls of radius  $n^{-1/d}$ .

### Proof of (3)

In Equation (28), let us choose  $w_n = \log(n) + c \log(\log(n))$  with  $c < (d-1)$  and introduce  $N_k = \lceil \exp(\sqrt{k}) \rceil$ , like in equation (3.12) in [12], we obtain for  $k$  large enough,  $\mathbb{P}(N_k V(\aleph_{N_k}) \leq w_{N_k}) \leq \exp(-\alpha_A k^{\frac{d-1-c}{2}}/2)$  and, Borel-Cantelli Lemma implies that, with probability one, for  $k$  large enough  $N_k V(\aleph_{N_k}) \leq w_{N_k}$ . The rest of the proof is exactly the same as in Lemma 5 in [12]

### Proof of (4):

For any  $u > 0$  let us introduce the sequence  $r_n = \left( \frac{\log n + (d+1+u) \log(\log(n))}{n} \right)^{1/d}$  and  $\varepsilon_n = \frac{1}{\log(n) \log(\log(n))}$ . Cover  $S$  with  $\nu_n \leq C_S \varepsilon_n^{-d} r_n^{-d}$  balls of radius  $\varepsilon_n r_n$ . Reasoning like in [12] we get

$$\mathbb{P} \left( \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} \geq d+1+u \right) = \mathbb{P}(\Delta_n \geq r_n) \leq \nu_n \left( 1 - r_n^d (1 - \varepsilon_n)^d (1 - 2K_f r_n \text{diam}(A)) \right)^n,$$

that implies,

$$\mathbb{P} \left( \frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} \geq d+1+u \right) \leq C_S \frac{(\log(\log(n)))^d}{(\log(n))^{2+u+o(1)}}.$$

From Borel-Cantelli Lemma, taking  $N_k = \lceil \exp(\sqrt{k}) \rceil$  it follows that, with probability one, for  $k$  large enough  $\frac{N_k V(\aleph_{N_k}) - \log(N_k)}{\log(\log(N_k))} \leq d+1+u$ . Now take  $n \geq N_k$  and  $n \leq N_{k+1}$  and suppose that  $\frac{N_k V(\aleph_{N_k}) - \log(N_k)}{\log(\log(N_k))} \leq d+1+u$ . We have,

$$nV(\aleph_n) \leq \frac{N_{k+1}}{N_k} V(\aleph_{N_k}) \leq \exp \left( \frac{1}{2\sqrt{k}} (1 + o(1)) \right) (\log(N_k) + (d+1+u) \log(\log(N_k))),$$

that entails,

$$nV(\aleph_n) \leq \left( 1 + \frac{1}{2\log(n)} (1 + o(1)) \right) (\log(n) + (d+1+u) \log(\log(n))).$$

Finally for  $n$  large enough,

$$nV(\aleph_n) \leq \log(n) + (d+1+u) \log(\log(n)) + 1,$$

so that

$$\frac{nV(\aleph_n) - \log(n)}{\log(\log(n))} \leq d + 1 + u + \frac{1}{\log(\log n)},$$

which concludes the proof of (4).

## 5 Appendix B

### 5.1 Proof of Theorem 3

The proof make use of the following two propositions. The first one gives conditions under which the maximal-spacing of two compact sets are close. The second one shows that if the set  $S$  is not convex, then  $\mathcal{R}(\mathcal{H}(S) \setminus S) > 0$ .

**Proposition 3.** *Let  $A$  and  $B$  be bounded and non-empty subsets of  $\mathbb{R}^d$ . If  $d_H(A, B) \leq \varepsilon$  and  $d_H(\partial A, \partial B) \leq \varepsilon$ . Then  $|\mathcal{R}(A) - \mathcal{R}(B)| \leq 2\varepsilon$ .*

*Proof.* First we introduce  $A' = \{x \in A : d(x, \partial A) > 2\varepsilon\}$  and prove that  $A' \subset B$  by contradiction. Suppose that there exists  $x \in A$  such that  $d(x, \partial A) > 2\varepsilon$  and  $x \notin B$ . Since  $d_H(A, B) \leq \varepsilon$  we have  $A \subset B^\varepsilon$ , then  $x \in B^\varepsilon \setminus B$ , so  $d(x, \partial B) \leq \varepsilon$ . Now, as  $d_H(\partial A, \partial B) \leq \varepsilon$ , by the triangular inequality,  $d(x, \partial A) \leq 2\varepsilon$ . which is a contradiction. From  $A' \subset B$  it follows that  $\mathcal{R}(A') \leq \mathcal{R}(B)$ . Now for all  $r < \mathcal{R}(A)$  there exist  $x \in A$  such that  $\mathcal{B}(x, r) \subset A$  so that  $\mathcal{B}(x, r - 2\varepsilon) \subset A'$  which entails  $\mathcal{R}(A') \geq \mathcal{R}(A) - 2\varepsilon$  and, finally,  $\mathcal{R}(B) \geq \mathcal{R}(A) - 2\varepsilon$ . Proceeding in the same way, we get  $\mathcal{R}(A) \geq \mathcal{R}(B) - 2\varepsilon$  that conclude the proof.  $\square$

**Proposition 4.** *Let  $S \subset \mathbb{R}^d$  be a non-convex, closed set with non-empty interior. Then,  $\mathcal{R}(\mathcal{H}(S) \setminus S) > 0$ .*

*Proof.* Since  $S$  is closed and non-convex there exists  $\bar{x} \in \mathcal{H}(S) \setminus S$  with  $d(x, S) = r > 0$ . By Corollary 7.1 in [10] we know that  $\mathcal{H}(S) = \mathcal{H}(\overset{\circ}{S})$  so that, for all  $\varepsilon > 0$ , there exists  $x_\varepsilon$  and  $\nu_\varepsilon > 0$  such that  $|x_\varepsilon - \bar{x}| \leq \varepsilon$  and  $\mathcal{B}(x_\varepsilon, \nu_\varepsilon) \subset \mathcal{H}(S)$ . Taking  $\varepsilon = r/2$  and  $\rho = \min(\nu_{r/2}, r/2) > 0$  we conclude that  $\mathcal{R}(\mathcal{H}(S) \setminus S) \geq \rho > 0$ .  $\square$

Now we can prove Theorem 3.

First observe that, if  $S$  is convex  $\mathcal{R}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \leq \mathcal{R}(S \setminus \aleph_n)$  and  $|\mathcal{H}(\aleph_n)| \leq |S|$  so that  $\tilde{V}_n \leq V(\aleph_n)$ . Then, from Corollary 1 we obtain that  $\mathbb{P}(\tilde{V}_n > c_{n,\gamma}) \leq \gamma + o(1)$ , and the test is asymptotically of level smaller than  $\gamma$ .

Now we prove that if  $S \in \mathcal{C}_P$ , then  $\mathbb{P}_{H_0}(\tilde{V}_n > c_{n,\gamma}) \rightarrow \gamma$ . Recall that, if  $S \subset \mathbb{R}^d$  is convex and  $\aleph_n = \{X_1, \dots, X_n\}$  is an iid random sample, uniformly drawn on  $S \in \mathcal{C}_P$ , in

[9] it is proved that, almost surely:

$$d_H(\mathcal{H}(\aleph_n), S) = \mathcal{O}\left((\log(n)/n)^{2/(d+1)}\right) \text{ and } d_H(\partial\mathcal{H}(\aleph_n), \partial S) = \mathcal{O}\left((\log(n)/n)^{2/(d+1)}\right). \quad (30)$$

Thus, by Proposition 3, we have that  $\left|\mathcal{R}(\mathcal{H}(\aleph_n) \setminus \aleph_n) - \mathcal{R}(S \setminus \aleph_n)\right| = \mathcal{O}\left((\log(n)/n)^{2/(d+1)}\right)$  almost surely. Therefore

$$\tilde{\Delta}_n(\aleph_n) = \frac{|\mathcal{H}(\aleph_n)|^{1/d}}{|S|^{1/d}} \left( \Delta(\aleph_n) + \mathcal{O}\left((\log(n)/n)^{2/(d+1)}\right) \right) \text{ a.s.}$$

The second equation in (30) also provides that  $|\mathcal{H}(\aleph_n)| = |S| + \mathcal{O}\left((\log(n)/n)^{2/(d+1)}\right)$  almost surely. Finally we have,

$$\tilde{V}_n = V(\aleph_n) \left( 1 + \mathcal{O}\left((\log(n)/n)^{2/(d+1)}\right) \right) \left( 1 + \mathcal{O}\left(\frac{(\log(n)/n)^{2/(d+1)}}{\Delta(\aleph_n)}\right) \right)^d \text{ a.s.}$$

Observe that from Corollary 1 ii) and iii)  $\Delta(\aleph_n) = (\log(n)/n)^{1/d} (1 + o(1))$  almost surely, then

$$\tilde{V}_n = V(\aleph_n) + \mathcal{O}\left((\log(n)/n)^{1+\frac{d-1}{d(d+1)}}\right) \text{ a.s.}$$

This, together with  $c_{n,\gamma} = \mathcal{O}(\log(n)/n)$  entails that  $\mathbb{P}(\tilde{V}_n \geq c_{n,\gamma}) \rightarrow \gamma$ , as desired.

To conclude the proof of the theorem consider now that  $S$  is not convex. First we prove that, if  $\varepsilon_n = d_H(S, \aleph_n)$ , then  $d_H(\mathcal{H}(S), \mathcal{H}(\aleph_n)) \leq 2\varepsilon_n$ . Indeed, for all  $x \in \mathcal{H}(S)$  there exist  $x_1 \in S$ ,  $x_2 \in S$  and  $\lambda \in [0, 1]$  such that  $x = \lambda x_1 + (1 - \lambda)x_2$ . Since  $\varepsilon_n = d_H(S, \aleph_n)$  there exist  $X_i \in \aleph_n$  and  $X_j \in \aleph_n$  such that  $\|x_1 - X_i\| \leq \varepsilon_n$  and  $\|x_2 - X_j\| \leq \varepsilon_n$  so that  $y = \lambda X_i + (1 - \lambda)X_j$  belongs to  $\mathcal{H}(\aleph_n)$ . By the triangular inequality  $\|x - y\| \leq 2\varepsilon_n$ . Since  $\mathcal{H}(\aleph_n) \subset \mathcal{H}(S)$  we also have that  $d_H(\partial\mathcal{H}(S), \partial\mathcal{H}(\aleph_n)) \leq 2\varepsilon_n$ , which implies that  $|\mathcal{H}(\aleph_n)| - |\mathcal{H}(S)| \leq \mathcal{O}(\varepsilon_n)$ . On the other hand we have  $d_H(\mathcal{H}(S) \setminus \aleph_n, \mathcal{H}(\aleph_n) \setminus \aleph_n) \leq 2\varepsilon_n$  and  $d_H(\partial(\mathcal{H}(S) \setminus \aleph_n), \partial(\mathcal{H}(\aleph_n) \setminus \aleph_n)) \leq 2\varepsilon_n$ . By Proposition 3 we get,  $\mathcal{R}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \geq \mathcal{R}(\mathcal{H}(S) \setminus \aleph_n) - 2\varepsilon_n$ . Since  $\mathcal{H}(S) \setminus S \subset \mathcal{H}(S) \setminus \aleph_n$  it follows,

$$\mathcal{R}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \geq \mathcal{R}(\mathcal{H}(S) \setminus S) - 2\varepsilon_n. \quad (31)$$

Since  $\varepsilon_n \rightarrow 0$  almost surely,  $|\mathcal{H}(\aleph_n)| \xrightarrow{\text{a.s.}} |\mathcal{H}(S)|$  and  $\mathcal{R}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \geq \mathcal{R}(\mathcal{H}(S) \setminus S)$  eventually almost surely. Then, there exists  $C_S$  a positive constant that depends on  $S$  such that,  $\tilde{V}_n \geq C_S$  eventually almost surely. Finally, since  $c_{n,\gamma} \rightarrow 0$  we conclude the proof.

## 5.2 Proof of Theorem 4

The proof of Theorem 4 is based on the following lemma



**Lemma 5.** Assume that the unknown density  $f$  fulfils condition  $B$  and  $S \in \mathcal{A}$ . Take  $K \in \mathcal{K}$ , and  $h_n = \mathcal{O}(n^{-\beta})$  with  $\beta \in (0, 1/d)$ .

Let  $\hat{f}_n(x)$  be the density estimator introduced in Definition 3. Then,

- (i) there exists a sequence  $\varepsilon_n^+$  such that  $\log(n)\varepsilon_n^+ \rightarrow 0$  and for all  $x \in S$ ,  $\left(\frac{f(x)}{\hat{f}_n(x)}\right)^{1/d} \geq 1 - \varepsilon_n^+$  e.a.s.
- (ii) there exist a sequence  $\varepsilon_n^- \rightarrow 0$  and a constant  $\lambda_0 > 0$  such that for all  $x \in \mathcal{H}(\mathbb{N}_n)$ ,  $(\hat{f}_n(x))^{1/d} \geq \lambda_0 - \varepsilon_n^-$  e.a.s.

*Proof.* We start the proof by establishing some useful preliminary results. First notice that, for  $S \in \mathcal{A}$ , with exactly the same kind of calculation we did to prove Lemma 2, choosing  $\rho_n = \left(\frac{4 \log n}{f_0 c_S \omega_d n}\right)^{1/d}$  we have,

$$\mathbb{P}(d_H(\mathbb{N}_n, S) \geq \rho_n) \leq C_S n^{-2} \quad \text{for } n \text{ large enough.} \quad (32)$$

Notice that, since  $K \in \mathcal{K}$ ,  $S \in \mathcal{A}$ , and  $K$  is bounded from below on a neighbourhood of the origin, there exist  $c_K'' > 0$  and  $r_K > 0$  such that,

$$\int_S K((u-x)/r) du \geq c_K'' r^d \quad \text{for all } x \in S \text{ and } r \leq r_K'. \quad (33)$$

We have, for all  $x \in S$ ,

$$\mathbb{E}f_n(x) = \int_{\{u: x+uh_n \in S\}} K(u) f(x+uh_n) du.$$

Using that  $f$  is Lipschitz and  $\int_{\mathbb{R}^d} K(u) du = 1$ , we get, for all  $x \in S$

$$\mathbb{E}f_n(x) \leq \int_{\{u: x+uh_n \in S\}} K(u) (f(x) + k_f \|u\| h_n) du \leq f(x) + k_f h_n c_K. \quad (34)$$

From (33) and the condition  $f(x) > f_0$  for all  $x \in S$ , it follows that,

$$\mathbb{E}f_n(x) \geq f_0 c_K' \quad \text{for all } x \in S. \quad (35)$$

We start by proving (i). The triangular inequality entails that,

$$\max_{x \in S} (\hat{f}_n(x) - f(x)) \leq \sup_{x \in S} |\hat{f}_n(x) - \mathbb{E}\hat{f}_n(x)| + \sup_{x \in S} (\mathbb{E}\hat{f}_n(x) - f(x)). \quad (36)$$

In order to deal with the first term on the right hand side of (36), observe that, since  $K \in \mathcal{K}$  and  $h_n = \mathcal{O}(n^{-\beta})$  with  $\beta \in (0, 1/d)$  we can apply Theorem 2.3 in [11]. Then, there exists a constant  $C_1$  such that, with probability one, for  $n$  large enough,

$$\sqrt{\frac{nh_n^d}{-\log(h_n)}} \sup_{x \in \mathbb{R}^d} |f_n(x) - \mathbb{E}f_n(x)| \leq C_1.$$

Thus

$$\sqrt{\frac{nh_n^d}{-\log(h_n)}} \sup_{x \in \mathfrak{N}_n} |f_n(x) - \mathbb{E}f_n(x)| \leq C_1,$$

and therefore,

$$\sqrt{\frac{nh_n^d}{-\log(h_n)}} \sup_{x \in S} |\hat{f}_n(x) - \mathbb{E}\hat{f}_n(x)| \leq C_1. \quad (37)$$

Now let us bound  $\sup_{x \in S} (\mathbb{E}\hat{f}_n(x) - f(x))$  from above. For all  $x \in S$  we have,

$$\begin{aligned} \mathbb{E}(\hat{f}_n(x)) &= \mathbb{E}(\hat{f}_n(x) | d_H(\mathfrak{N}_n, S) \leq \rho_n) \mathbb{P}(d_H(\mathfrak{N}_n, S) \leq \rho_n) + \\ &\quad \mathbb{E}(\hat{f}_n(x) | d_H(\mathfrak{N}_n, S) > \rho_n) \mathbb{P}(d_H(\mathfrak{N}_n, S) > \rho_n). \end{aligned} \quad (38)$$

Since  $\{(x, y) \in S^2, \|x - y\| \leq h_n\}$  is compact, the Lebesgue dominate convergence theorem entails that there exist  $y_0 \in S$  such that  $\|x - y_0\| \leq \rho_n$ , and a sequence  $y_k$  with  $y_k \rightarrow y_0$ ,  $\|y_k - y_0\| \leq \rho_n$ , such that for  $n$  large enough, with probability one

$$\begin{aligned} \mathbb{E}(\hat{f}_n(x) | d_H(S, \mathfrak{N}_n) \leq \rho_n) &\leq \sup_{x \in S} \mathbb{E} \left( \limsup_{y \in S: \|x-y\| \leq \rho_n} f_n(y) \right) = \\ &\sup_{x \in S} \mathbb{E} \left( \lim_{y_k \rightarrow y_0} f_n(y_k) \right) = \sup_{x \in S} \lim_{y_k \rightarrow y_0} (\mathbb{E}(f_n(y_k))) \leq \sup_{x \in S} \sup_{y \in S: \|x-y\| \leq \rho_n} \mathbb{E}(f_n(y)). \end{aligned}$$

Now applying (34) and the Lipschitz continuity of  $f$  we obtain,

$$\mathbb{E}(\hat{f}_n(x) | d_H(S, \mathfrak{N}_n) \leq \rho_n) \leq \max_{y \in S, \|x-y\| \leq \rho_n} \{f(y) + k_f h_n c_K\} \leq f(x) + k_f \rho_n + k_f h_n c_K. \quad (39)$$

With the same kind of argument it can be proved that,

$$\mathbb{E}(\hat{f}_n(x) | d_H(\mathfrak{N}_n, S) \geq \rho_n) \leq \sup_{y \in S} \mathbb{E}f_n(y) \leq f_1 + k_f h_n c_K. \quad (40)$$

From equations (38), (39), (40) and (32) we get,

$$\sup_{x \in S} (\mathbb{E}(\hat{f}_n(x)) - f(x)) \leq k_f \rho_n + k_f h_n c_K + (f_1 + k_f h_n c_K) C_S n^{-2}. \quad (41)$$

Take now  $\varepsilon_n = k_f \rho_n + k_f h_n c_K + (f_1 + k_f h_n c_K) C_S n^{-2} + C_1 \left( \frac{nh_n^d}{-\log(h_n)} \right)^{1/2}$ . Which fulfils  $\log(n)\varepsilon_n \rightarrow 0$ . From equations (36), (37), (41), we obtain that, with probability one, for  $n$  large enough,

$$\max_{x \in S} (\hat{f}_n(x) - f(x)) \leq \varepsilon_n.$$

Then, for all  $x \in S$ ,  $\hat{f}_n(x) - f(x) \leq f(x)\varepsilon_n/f_0$ , and thus,  $\frac{\hat{f}_n(x)}{f(x)} \leq 1 + \frac{\varepsilon_n}{f_0}$ , or equivalently,

$$\left( \frac{f(x)}{\hat{f}_n(x)} \right)^{1/d} \geq \left( 1 + \frac{\varepsilon_n}{f_0} \right)^{-1/d}.$$

Finally, taking  $\varepsilon_n^+ = (1 - (1 + \varepsilon_n/f_0)^{-1/d}) \sim \varepsilon_n/(df_0)$  (observe that we have  $\varepsilon_n^+ \log(n) \rightarrow 0$ ) then  $\max_{x \in S} \left( \frac{f(x)}{\hat{f}_n(x)} \right)^{1/d} \geq 1 - \varepsilon_n^+$  eventually almost surely, which concludes the proof of (i).

In order to prove (ii), observe that

$$\min_{x \in \mathbb{R}^d} \hat{f}_n(x) \geq \min_{x \in \mathbb{R}^d} \mathbb{E} \hat{f}_n(x) - \max_{x \in \mathbb{R}^d} |\mathbb{E} \hat{f}_n(x) - \hat{f}_n(x)|.$$

Since we have already proved that  $\max_{x \in \mathbb{R}^d} |\mathbb{E} \hat{f}_n(x) - \hat{f}_n(x)| \rightarrow 0$  a.s., it remains to prove that  $\min_{x \in \mathbb{R}^d} \mathbb{E} \hat{f}_n(x)$  is bounded from below by a positive constant. From  $\min_{x \in \mathbb{R}^d} \mathbb{E} \hat{f}_n(x) = \min_{x \in \aleph_n} \mathbb{E} f_n(x)$  and (35), we get

$$\min_{x \in \mathbb{R}^d} \mathbb{E} \hat{f}_n(x) \geq \min_{x \in S} \mathbb{E} f_n(x) \geq f_0 c'_K.$$

□

Now we are ready to prove Theorem 4 .

### Proof of Theorem 4 a)

Recall that

$$\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) = \sup \left\{ r : \exists x \text{ such that } x + \frac{r}{\hat{f}_n(x)^{1/d}} A \subset \mathcal{H}(\aleph_n) \setminus \aleph_n \right\},$$

with  $A$  the ball  $\mathcal{B}(O, \omega_d^{-1/d})$ .

Under  $H_0$  ( $S$  is convex),

$$\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \leq \sup \left\{ r : \exists x \text{ such that } x + \frac{r}{\hat{f}_n(x)^{1/d}} A \subset S \setminus \aleph_n \right\}.$$

If we apply Lemma 5 (i), (notice that all convex sets are in  $\mathcal{A}$ ) we get,

$$\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \leq \sup \left\{ r : \exists x \text{ such that } x + \frac{r}{f(x)^{1/d}} (1 - \varepsilon_n^+) A \subset S \setminus \aleph_n \right\}.$$

Equivalently,  $\Delta(\aleph_n) \geq (1 - \varepsilon_n^+) \hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n)$ , and therefore  $\mathbb{P}(\hat{V}_n \geq c_{n,\gamma}) \leq \mathbb{P}(V(\aleph_n) \geq (1 - \varepsilon_n^+)^d c_{n,\gamma})$ , from where it follows that  $\mathbb{P}(\hat{V}_n \geq c_{n,\gamma})$  can be majorized by,

$$\mathbb{P}\left(U(\aleph_n) \geq -(1 - \varepsilon_n^+)^d \log(-\log(1 - \gamma)) + ((1 - \varepsilon_n^+)^d - 1)(\log(n) + (d-1)\log(\log(n)) + \log(\alpha_{\mathcal{B}}))\right).$$

Therefore, by Theorem 2, (using that  $\log(n)\varepsilon_n^+ \rightarrow 0$ ) we get that,

$$\mathbb{P}(\hat{V}_n \geq c_{n,\gamma}) \leq \mathbb{P}(U(\aleph_n) \geq -\log(-\log(1 - \gamma)) + o(1)) \rightarrow \gamma.$$

## Proof of Theorem 4 b)

From Lemma 5 (ii), we have that

$$\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \geq (\lambda_0 - \varepsilon_n^-) \mathcal{R}(\mathcal{H}(\aleph_n) \setminus \aleph_n),$$

where  $\varepsilon_n^- \rightarrow 0$  a.s. Then, under  $H_1$  ( $S$  is not convex), from (31), we obtain

$$\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \geq (\lambda_0 - \varepsilon_n^-) (\mathcal{R}(\mathcal{H}(S) \setminus S) - 2d_H(S, \aleph_n)).$$

Since  $S \in \mathcal{A}$ ,  $d_H(S, \aleph_n) \rightarrow 0$  a.s. (see [4]), and  $\mathcal{R}(\mathcal{H}(S) \setminus S) > 0$  (see Proposition 4) then with probability one, for  $n$  large enough

$$\hat{\delta}(\mathcal{H}(\aleph_n) \setminus \aleph_n) \geq \frac{1}{2} \lambda_0 \mathcal{R}(\mathcal{H}(S) \setminus S),$$

and Theorem 4 b) follows from the fact that  $c_{n,\gamma} \rightarrow 0$ .

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